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Research Article

Self Generating n-Tuples

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ABSTRACT:
Background: The Pythagorean triple based on Pythagorean Theorem, were known in to ancient Babylon and Egypt. The interrelation of the three was known as far back as thousands of years, but it was Pythagoras who explicitly explained their equation.
Purpose: Different methods have been put forth by the mathematicians for generation of Pythagorean's triple and n-tuples but this paper provides a unique method how these get self-generated by use of simple algebraic expansions.
Methods: An algebraic quantity $(a+b)$ squared equals to $(a-b)$ squared plus $4ab$ and if a or b is assigned such a value that makes $4ab$ a whole square, then $(a+b)$, $(a-b)$ and under root of $4ab$ turns Pythagorean's triple. Similarly, utilizing such algebraic identities, Pythagorean's quadruple up to n-tuples can be generated. If $(a+b)$ is squared, it provides a squared plus b squared plus $2ab$. If quantity $2ab$ is transformed to a whole square on account of assigning values to a or b, then Pythagorean's quadruples are obtained.
Results: Assigning specific values to the terms of simple algebraic identities results in the generation of Pythagorean triples and n-tuples.
Conclusions: This paper presents empirical research in which algebraic identities are utilized, resulting in the self-generation of Pythagorean n-tuples. Specific formulas need not be applied, as basic algebraic identities are well known to scholars and students alike.

KEYWORDS: Pythagorean's triples, Quadruples, Quintuples, n-tuples
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1. Introduction

In a right angled triangle, if a is the length of its base AB, b of perpendicular AC, then c length of its hypotenuses BC is given by the relation $c^2 = a^2 + b^2$ and a, b, c are called Pythagorean's triple [?]. Euclid gave a formula to generate these triples using two integers m and n [?]. In his formula, referring to Figure 1, hypotenuse c equals to $(m^2 + n^2)$, base a to $(2mn)$ and perpendicular b to $(m^2 - n^2)$. Since quantities a, b, c appear in squares, it does not matter whether these are positive or negative in polarity. When triple a, b, c do not have common factors, it is primitive otherwise non-primitive. Further, if in a right angled triangle ABC, a perpendicular of length d is raised at point C on its hypotenuse BC up to point D, then length of closing side BD of quadrilateral $BCAD$, is given by the relation $e^2 = c^2 + d^2 = a^2 + b^2 + d^2$ and integers a, b, d, e are called Pythagorean's quadruple. Kindly refer to Figure 1. If a is odd and m, n, p, q are positive integers with greatest common divisor 1, the quadruple a, b, d, e can be generated by formulae $a = m^2+n^2-p^2-q^2$, $b = 2(mq+np)$,$d = 2(nq-mp)$ and $e = m^2+n^2-p^2-q^2$,$b = 2(mq+np)$, $d = 2(nq-mp)$ and $e = m^2+n^2+p^2+q^2$ provided $m+n+p+q$ is odd [? ? ?]. Therefore, a primitive quadruple is given by identity

$$(m^2 + n^2 + p^2 + q^2)^2 = (m^2 + n^2 - p^2 - q^2)^2 + 2(mq + np)^2 + 2(nq - mp)^2, \quad (1)$$

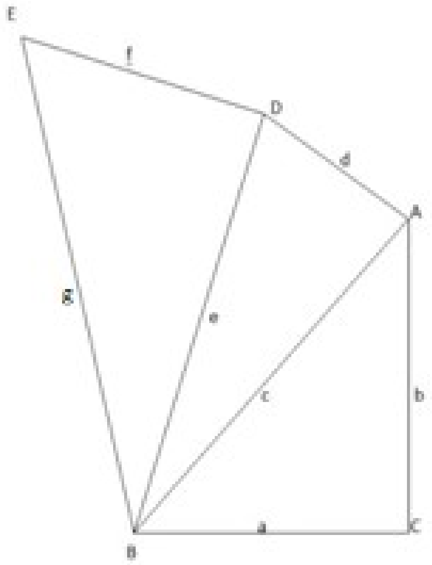

Figure 1: Displaying Sides BA, AC, CB Pythagorean's Triple, BD, DA, AC, CB Quadruple, AE, ED, DA, AC, CB Quintuple.

According to Paul Oliverio, if a and b are of opposite parity, then $d = (a^2+b^2-1)/2$ and $e = (a^2+b^2+1)/2$, satisfy the relation $e^2 = a^2 + b^2 + d^2$. On the other hand, if a and b are even, then $d = \frac{(a^2+b^2)}{4} - 1$ and $e = \frac{(a^2+b^2)}{4} + 1$ satisfy the relation $e^2 = a^2 + b^2 + d^2$ [**?**]. Coming to quintuple, if another perpendicular of length f is drawn at point D on side BD up to point E, then closing side BE of length g of polygon of five sides, is given by relation $g^2 = a^2 + b^2 + d^2 + f^2$ and a, b, d, f, b, g are called Pythagorean's quintuple. Kindly refer to Figure 1. proceeding in this fashion, n^{th} side an of polygon of n sides, can be calculated by relation

$$a_n^2 = a_1^2 + a_2^2 + a_3^2 + \ldots\ldots + a_{n-1}^2$$

where lengths in integers $a_1, a_2, a_3, \ldots, a_n$ of the sides of the polygon are given the nomenclature of Pythagorean's n-tuple. With this background, I have determined formulae in proceeding sections for generating triples to n-tuples using elementary algebraic identities. I have adopted four different un-attempted methods for generation of such tuples.

2. Method & Materials

2.1. *Self-Generating Quadruples*

Consider an algebraic identity

$$(Ax + Ba)^2 = (Ax)^2 + 2ABxa + (Ba)^2 \qquad (2)$$

Each terms in right hand side of this identity, will be a square, when term $(2ABxa)$ is also a square. That necessitates, both x and a must be squares and also either A should be double of B or B should be double of A. Substituting B with $2A$, x with x^2 and a with a^2, Equation (2), on simplification, takes the form,

$$(x^2 + 2a^2)^2 = (x^2)^2 + (2xa)^2 + (2a^2)^2 \qquad (3)$$

and, when $A = 2B$, Equation (2) takes the form

$$(2x^2 + a^2)^2 = (2x^2)^2 + (2xa)^2 + (a^2)^2 \qquad (4)$$

Where x is a variable. These are, in fact, parametric solutions to Pythogorean's quadruples for all real rational values of x and a. Derivation of Equations(3) and (4) proves LEMMAS 1 and 2.

LEMMA 1:
(x^2), $(2xa)$, $(2a^2)$ and $(x^2 + 2a^2)$ are always Pythagorean's quadruple for all integer values of x and a.

LEMMA 2:
$(2x^2)$, $(2xa)$, (a^2), and $(2x^2 + a^2)$ are always Pythagorean's quadruple for all integer values of x and a.

Examples:
When (x, a) have values (1, 3), (5, 3), (11, 1), (7, 3), using equation (3) quadruples obtained (1, 6, 18, 19), (25, 30, 18, 43), (121, 22, 2, 123), (98,42, 9, 107) respectively.

2.1.1. *Self-Generating n-Tuples*

If x is substituted with x^2, a with a^2, A and B each with 1 in Equation (2), then $(x^2 + a^2)^2$, on expansion equals $(x^2)^2 + (xa)^2 + (a^2)^2$. This can also be written,

$$\left\{x^2\left(\frac{1^2+1^2}{2}\right) + a^2\right\}^2 = (x^2)^2 + (xa)^2(1^2+1^2) + (a^2)^2$$

In general, we can write

$$\left\{x^2\left(\frac{a_1^2+a_2^2}{2}\right)+a^2\right\}^2 =$$
$$\left\{x^2\left(\frac{a_1^2+a_2^2}{2}\right)\right\}^2 + (xa)^2(a_1^2+a_2^2)+(a^2)^2.$$

On multiplying with 4,

$$[x^2(a_1^2+a_2^2)+2a^2)]^2 = [x^2(a_1^2+a_2^2)]^2+[2a_1xa]^2+[2a_2xa]^2+[2a^2]^2 \quad (5)$$

Similarly we can derive

$$[2x^2+a^2(a_1^2+a_2^2)]^2 = [2x^2]^2+[2a_1xa]^2+[2a_2xa]^2+[(a_1^2+a_2^2)a^2]^2 \quad (6)$$

where x, a_2, a_1 and a are real rational quantities. These are parametric solutions to quintuple. Derivation of Equations (5) and (6) proves LEMMA 3 and 4.

LEMMA 3:

$\{x^2(a_1^2+a_2^2)\}, (2a_1xa), (2a_2xa), (2a^2)$ and $\{x^2(a_1^2+a_2^2)+a_2^2\}$

are always Pythagorean's quintuple for all integer values of x, a_1, a_2 and a.

LEMMA 4:

$(2x^2), (2a_1xa), (2a_2xa), \{a^2(a_1^2+a_2^2)\}$ and $\{2x^2+a^2(a_1^2+a_2^2)\}$

are always Pythagorean's quintuple for all integer values of x, a_1, a_2 and a.

Examples:

When (x, a, a_1, a_2) have values (2, 3, 4, 5), (3, 2 ,5, 6), (7, 5, 3, 1), (10, 1, 2, 3) using Equation (5) we obtain quintuples (164, 48, 60, 18, 182), (549, 60, 72, 8, 557), (490, 210,70, 50, 540), (1300, 40, 60, 2, 1302) respectively and using Equation (6), we obtain quintuple (8, 48, 60, 369, 377), (18, 60, 72, 244, 262), (98, 210,70, 250, 348), (200, 40, 60, 13, 213) respectively. When $n = 6$ we substitute $(a_1^2+a_2^2)$ with $(a_1^2+a_2^2+a_3^2)$ in Equation (5) and (6) and we obtain parametric solutions to sextuples,

$$[x^2(a_1^2+a_2^2+a_3^2)+2a^2]^2 = [x^2(a_1^2+a_2^2+a_3^2)]^2+[2a_1xa]^2+[2a_2xa]^2 +[2a_3xa]^2+[2a^2]^2 \quad (7)$$

and

$$[2x^2+(a_1^2+a_2^2+a_3^2)a^2]^2 = [2x^2]^2+[2a_1xa]^2+[2a_2xa]^2+[2a_3xa]^2+ [(a_1^2+a_2^2+a_3^2)a^2]^2 \quad (8)$$

respectively. Derivation of these Equations proves LEMMAS 5 and 6.

LEMMA 5:

$\{x^2(a_1^2+a_2^2+a_3^2)\}$,$(2a_1xa)$,$(2a_2xa)$, $(2a_3xa)$,$(2a^2)$ and $\{x^2(a_1^2+a_2^2+a_3^2)+2a^2\}$ are always Pythagorean's sextuples for all integer values of x, a_1, a_2, a_3 and a.

LEMMA 6:

$(2x^2), (2a_1xa), (2a_2xa), (2a_3xa), \{a^2(a_1^2+a_2^2+a_3^2)\}$ and $\{2x^2+a^2(a_1^2+a_2^2+a_3^2)\}$ are always Pythagorean's sextuples for all integer values of x, a_1, a_2, a_3 and a.

Examples:

When (x, a, a_1, a_2, a_3) are equal to (2, 3, 4, 5, 6), (3, 2, 5, 6, 7), (7, 5, 3, 1, 2), (10, 1, 2, 3, 4), using Equation (7), we obtain Pythagorean's sextuples, (308, 48, 60, 72, 18, 326), (990, 60, 72, 84, 8, 998), (686, 210, 70, 140, 50, 736), (2900, 40, 60, 80, 2, 2902) respectively and using Equation (8), we obtain Pythagorean's sextuples respectively. Generalizing the equations for n-tuples,

$$[x^2(a_1^2+a_2^2+a_3+\ldots+a_{n-3})+2a^2]^2 = [x^2(a_1^2+a_2^2+a_3+\ldots+a_{n-3})]^2 +[2a_1(xa)]^2+[2a_2(xa)]^2+[2a_3(xa)]^2 +\ldots+[2a_{n-3}(xa)]^2+[2a^2]^2. \quad (9)$$

and

$$[2x^2+a^2(a_1^2+a_2^2+a_3^2+\ldots+a_{n-3})]^2 = [a^2(a_1^2+a_2^2+a_3^2+\ldots+a_{n-3})]^2 +[2a_1(xa)]^2+[2a_2(xa)]^2+[2a_3(xa)]^2 +\ldots+[2a_{n-3}(xa)]^2+[2x^2]^2, \quad (10)$$

where $x, a, a_1, a_2, a_3, ..., a_{n-3}$ are integers, if $S_{n-3} = a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2$, then

$$\begin{aligned}[x^2 S_{n-3} + 2a^2]^2 = \\ [x^2 S_{n-3}]^2 + [2a_1(xa)]^2 + [2a_2(xa)]^2 \\ +[2a_3(xa)]^2 + ... + [2a_{n-3}(xa)]^2 \\ +[2a^2]^2.\end{aligned} \tag{11}$$

and

$$\begin{aligned}[2x^2 + S_{n-3}a^2]^2 = \\ [2x^2]^2 + [2a_1(xa)]^2 + [2a_2(xa)]^2 \\ +[2a_3(xa)]^2 + ... + [2a_{n-3}(xa)]^2 \\ +[S_{n-3}a^2]^2.\end{aligned} \tag{12}$$

Derivation of Equations (9) and (10) proves LEMMA 7 and LEMMA 8.

LEMMA 7:

$[2a_1(xa)]$, $[2a_2(xa)]$, $[2a_3(xa)]$,...,$[2a_{n-3}(xa)]$, $[2a^2]$, $[x^2 S_{n-3}]$ and $[x^2 S_{n-3} + 2a^2]$ are always Pythagorean's n-tuples for all real rational values of $a_1, a_2, a_3, ..., a_{n-3}, a$ and x, where,
$S_{n-3} = a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2$.

LEMMA 8:

$[2a_1(xa)]$, $[2a_2(xa)]$, $[2a_3(xa)]$,...,$[2a_{n-3}(xa)]$, $[2x^2]$, $[a^2 S_{n-3}]$ and $[2x^2 + S_{n-3}a^2]$ are always Pythagorean's tuples for all real rational values of $a_1, a_2, a_3, ..., a_{n-3}, a$ and x, where,
$S_{n-3} = a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2$.

Examples:

When $(x, a, a_1, a_2, a_3, a_4, a_5, a_6)$,$(x, a, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ are equal to (1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 1, 5, 6, 7, 8, 9, 10, 4, 1), then using Equation (11), we obtain Pythagorean's n-tuples. (12, 16, 20, 24, 28, 32, 8, 199, 207), (6, 30, 36, 42, 48, 54, 60, 24, 6, 18, 373, 391) respectively and when $(x, a, a_1, a_2, a_3, a_4, a_5, a_6)$, $(x, a, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ are equal to (2, 3, 4, 5, 6, 7, 8), (1, 3, 1, 5, 6, 7, 8, 9, 10, 4, 1), then using Equation (12) we obtain Pythagorean's n-tuples (36, 48, 60, 72, 84, 96, 8, 1791, 1799), (6, 30, 36, 42, 48, 54, 60, 24, 6, 2, 3357, 3359) respectively.

2.1.2. n-Tuples with Fixed Terms $a_1, a_2, a_3, ..., a_{n-3}$

If my requirement is such that the tuples $a_1, a_2, a_3, ..., a_{n-3}$ must appear in n-tuples, then sum must be even integer and that requires, odd integers, if any, out of $a_1, a_2, a_3, ..., a_{n-3}$ must occur in pairs.
If, $2s_{n-3} = S_{n-3}$, $x = 1$
and $a = 1$,
then Equation (11) and (12) can be

$$\begin{aligned}[S_{n-3} + 1]^2 = \\ [S_{n-3}]^2 + [a_1]^2 + [a_2]^2 + [a_3]^2 \\ +... + [a_{n-3}]^2 + [1]^2\end{aligned} \tag{13}$$

Kindly note the difference between s and S, both should not be confused denoting to same quantity. These derivations prove LEMMA 9.

LEMMA 9:

$[a_1]$,$[a_2]$,$[a_3]$,...$[a_{n-3}]$ will always be part of Pythagorean's n-tuples with other three terms [**?**], $[s_{n-3}]$, $[s_{n-3} + 1]$ where $a_1, a_2, a_3, ..., a_{n-3}, s_{n-3}$ are integers, if

$$s_{n-3} = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2)$$

and

$$(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2)$$

are even integer.

Examples:

Ages of a family members in years, are 67, 59, 36 and 30. Their sum being even integer, these can be expressed in Pythagoras n-tuple where a_1, a_2, a_3 and a_4 are 67, 59 , 36 and 30 respectively and $s_4 = (1/2)(10116) = 5083$. Using Equation (13), 50842 = 50832 + 12 + (672 + 592 + 362 + 302). Given in the bracket are ages of the family members expressed in Pythagorean's n-tuple. If age 2 years of the pet is included, then $s_5 = (1/2)(10120) = 5085$ and the corresponding n-tuple is 50862 = 50852+12+(672+592+362+302+22). Seven is the lucky number for the family and 563 is the house number, on including these, we get, $s_7 = (1/2)(327188) = 163594$. Corresponding n-tuple is 1635952 = 1635942 + 12 + (672 + 592 + 362 + 302 + 22 + 72 + 5632).

2.2. Second Method to Generate Pythagorean's n-Tuples

Let a, b, and c be Pythagorean's triple satisfying relation $c^2 = a^2 + b^2$ [?]. We express integers c and b in algebraic form $c_1x + c_2$ and $b_1x + b_2$ respectively, where c_1, c_2, b_1 and b_2 are real rational quantities, then, $(c_1x+c_2)^2 - (b_1x+b_2)^2 = a^2$. On expansion and rearranging, $x^2(c_1^2 - b_1^2) + 2x(c_1c_2 - b_1b_2) + c_2^2 - b_2^2 = a^2$. Let, $c_1 = b_1 = c$ and, $c_2 = -b_2 = p^2$,
then $(cx+p^2)^2-(cx-p^2)^2 = 4xp^2c = a^2$. Substituting y_2 for x, above equation gets transformed into

$$(cy^2 + p^2)^2 - (cy^2 - p^2)^2 = 4p^2y^2c = a^2. \quad (14)$$

Let, $c = S_{n-2} = a_1^2 + a_2^2 + a_3^2 + ... + a_{n-2}^2$, then Equation (14) takes the form ;

$$(y^2S_{n-2}+p^2)^2 = (y^2S_{n-2}-p^2)^2+4p^2y^2(a_1^2+a_2^2+a_3^2 +.... + a_{n-2}^2.$$

When $a_2 = a_3 = a_4 = ... = a_{n-2} = 0$, then this equation will be a parametric solution to triples, otherwise, it can be written in S form as;

$$(y^2S_{n-2} + p^2)^2 = (y^2S_{n-2} - p^2)^2 + (2pya_1)^2 + (2pya_2)^2 +(2pya_3)^2 + ... + (2pya_{n-2})^2, \quad (15)$$

which is a parametric solution to n-tuples. Equation (15) can also be transformed into,

$$(y^2 + S_{n-2}p^2)^2 = (y^2 - S_{n-2}p^2)^2 + (2pya_1)^2 + (2pya_2)^2 +(2pya_3)^2 + ... + (2pya_{n-2})^2. \quad (16)$$

which is a parametric solution to n-tuples. Putting $p = 1$, Equation (13) takes the form,

$$(y^2 + S_{n-2})^2 = (y^2 - S_{n-2})^2 + (2ya_1)^2 + (2ya_2)^2 + (2ya_3)^2 +... + (2ya_{n-2})^2. \quad (17)$$

For triples, $n = 3, S_1$ equals to a_1^2 and parametric solution is

$$(y^2 + a_1^2)^2 = (y^2 - a_1^2)^2 + (2ya_1)^2. \quad (18)$$

For quadruples, n=4,S_2 equals to $a_1^2 + a_2^2$ and parametric solution is,

$$(y^2 + a_1^2 + a_2^2)^2 = (y^2 - a_1^2 - a_2^2)^2 + (2ya_1)^2 + (2ya_2)^2. \quad (19)$$

These derivations prove LEMMA 10.

LEMMA 10:
When $y, p, a_1, a_2, a_3, ..., a_{n-2}$ are integers and $S_{n-2} = a_1^2 + a_2^2 + a_3^2 + ... + a_{n-2}^2$ then, $(y^2 - S_{n-2}p^2), (2pya_1), (2pya_2), (2pya_3), ...,(2pya_{n-2})$, and,$(y^2+S_{n-2}p^2)$ are tuples of Pythagorean's n-tuples.

Examples:
For Pythagorean's quadruples, we assume (y, a_1, a_2) as (4, 1, 2), (4, 2, 3), (17, 1, 9), then using Equation (19) corresponding quadruples are (11, 8, 16, 21), (3, 16, 24, 29), (297, 34, 306, 371) respectively. For $n = 9$, assuming $(y, a_1, a_2, a_3, ..., a_7)$ as (41, 2, 4, 6, 3, 1, 9, 11) and using Equation (17), we obtain corresponding n-tuple (1413, 164, 328, 492, 246, 82, 738, 902, 1949).

2.2.1. n-Tuples with Fixed Terms a_1, a_2, a_3, ..., a_{n-2}

I consider Equation (17) and substitute S_{n-2} with $S_{n-2}/4$ and y with 1, then

$$(1 + S_{n-2}/4)^2 = (1 - S_{n-2}/4)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 +... + (a_{n-2})^2.$$

Since $S_{n-2}/4$ should have integer value, that requires, it should be of form 4k. To achieve this objective, there must be, at least, two pairs of odd integers, if $a_1, a_2, a_3, ..., a_{n-2}$ happen to have odd integers. In that case, let $S_{n-2} = 4s_{n-2}$, then

$$(1 + s_{n-2})^2 = (1 - s_{n-2})^2 + (a_1^2 + a_2^2 + a_3^2 + ... + a_{n-2}^2). \quad (20)$$

Examples:
Let $a_1 = 1001, a_2 = 1, a_3 = 10001, a_4 = 11$ and $a_5 = 64$, then $s_5 = 25256555$ and n-tuples, where $n = 7$, are given by relation,
252565562 = 252565542 + 10012 + 12 + 100012 + 112 + 642. Let $a_1 = 16, a_2 = 78, a_3 = 1248$ and $a_4 = 2$, then $s_4 = 390962$ and n-tuples are given by relation, 3909632 = 3909612+162+782+12482+22.

2.3. Third Method To Generate n-Tuples Using Recursive Relation

It is proved in Equation (18) that

$$(x_1^2+a^2)^2=(x_1^2-a^2)^2+(2x_1a^2)^2$$

On putting, $x_1=x_2^2+a^2$ and substituting $(x_2^2+a^2)^2$ with $(x_2^2-a^2)^2+2^2a^2(x_2)^2$, I get,

$$\begin{aligned}\{(x_2^2+a^2)^2+a^2\}^2 &=\\ \{(x_2^2+a^2)^2-a^2\}^2+\{2a(x_2^2-a^2)\}^2&\\ +\{2^2a^2x_2\}^2.&\end{aligned} \tag{21}$$

Equation (21) is a parametric solution to quadruples and proves LEMMA 11.

LEMMA 11:
$\{(x_2^2+a^2)^2-a^2\}, \{2a(x_2^2-a^2)\}, \{2^2a^2(x_2)\}$and $\{(x_2^2+a^2)^2+a^2\}$,

are always Pythagorean's quadruples, where x_2 and a are integers.

Example:

On assigning to (x_2,a) values of (5, 1), (1, 3), (7, 3), (5, 4), Pythagorean's quadruples obtained are (675, 48, 20, 677), (91, 48, 36, 109), (3355, 240, 252, 3373), (1665, 72, 320, 1697) respectively. On putting $x_2=x_3^2+a^2$ and substituting $(x_3^2+a^2)^2$ with $(x_3^2-a^2)^2+2^2a^2(x_3)^2$, we get,

$$\begin{aligned}[\{(x_3^2+a^2)^2+a^2\}^2+a^2]^2 &=\\ [\{(x_3^2+a^2)^2+a^2\}^2-a^2]^2&\\ +[2a\{(x_3^2+a^2)^2-a^2\}&\\ +[2^2a^2(x_3^2-a^2)]^2+[2^3a^3x_3]^2.&\end{aligned} \tag{22}$$

Also Equation (22) is a parametric solution to quintuples and proves LEMMA 12.

LEMMA 12:
$[\{(x_2^2+a^2)^2+a^2\}^2-a^2], [2a\{(x_3^2+a^2)^2-a^2\}], [2^2a^2(x_3^2-a^2)], [2^3a^3(x_3)]and[\{(x_3^2+a^2)^2+a^2\}^2+a^2]$.

are always Pythagorean's quintuples, where x_3 and a are integers.

Examples:

On assigning to (x_3,a) values of (1, 2), (3, 2), (4, 1) Pythagorean's quintuples obtained are (835, 84, 48, 64, 845), (29925, 660, 80, 192, 29933), (84099, 576, 60, 32, 84101) respectively. On putting $x_3=x_4^2+a^2$ and substituting $(x_4^2+a^2)^2$ with $(x_4^2-a^2)^2+2^2a^2(x_4)^2$, we get,

$$\begin{aligned}[[\{(x_4^2+a^2)^2+a^2\}^2+a^2]^2+a^2]^2 &=\\ [[\{(x_4^2+a^2)^2+a^2\}^2+a^2]^2-a^2]^2&\\ +[2a[\{(x_4^2+a^2)^2+a^2\}^2-a^2]]^2&\\ +[2^2a^2(x_4^2+a^2)^2-a^2\}]^27&\\ +[2^3a^3(x_4^2-a^2)]^2+[2^4a^4(x_4)]^2.&\end{aligned} \tag{23}$$

Also Equation (23) is a parametric solution to sextuples and proves LEMMA 13.

LEMMA 13:
$[[\{(x_4^2+a^2)^2+a^2\}^2+a^2]^2-a^2], [2a[\{(x_4^2+a^2)^2+a^2\}^2-a^2]], [2^2a^2(x_4^2+a^2)^2-a^2\}]^2, [2^3a^3(x_4^2-a^2)]^2+[2^4a^4(x_4)]^2, [[\{(x_4^2+a^2)^2+a^2\}^2+a^2]^2+a^2]^2.$

are always Pythagorean's sextuples, where x_4 and a are integers.

Examples:

On assigning to (x_4,a) values of (1, 3), (1, 2), (3, 1), (2, 3), Pythagorean's sextuples obtained are (141372091, 71232, 3276, 1728, 1296, 141372109), (714021, 3348, 336, 192, 256, 714029) respectively. Proceeding in the way, I determined parametric solution to n-tuples,

$$\begin{aligned}[[\{(x_{n-2}^2+a^2)^2+a^2\}^2+a^2]^2+\ldots+a^2]^2 &=\\ [[[\{(x_{n-2}^2+a^2)^2+a^2\}^2+a^2]^2+\ldots-a^2]^2&\\ +[2a[\{(x_{n-2}^2+a^2)^2+a^2\}^2+\ldots..-a^2]]^2&\\ +[2^2a^2\{(x_{n-2}^2+a^2)^2+\ldots.-a^2\}]^2+\ldots.&\\ +[2^{(n-3)}a^{(n-3)}(x_{n-2}^2-a^2)]^2&\\ +[2^{(n-2)}a^{(n-2)}(x_{n-2})]^2.&\end{aligned} \tag{24}$$

Also Equation (24) proves LEMMA 14.

LEMMA 14:
$[[\{(x_{n-2}^2+a^2)^2+a^2\}^2+a^2]^2+\ldots-a^2]^2, [2a[\{(x_{n-2}^2+a^2)^2+a^2\}^2+\ldots..-a^2]]^2, [2^2a^2\{(x_{n-2}^2+a^2)^2+\ldots.-a^2\}]^2, \ldots, [2^{(n-3)}a^{(n-3)}(x_{n-2}^2-a^2)]^2,$
$[2^{(n-2)}a^{(n-2)}(x_{n-2})]^2,$
$[[\{(x_{n-2}^2+a^2)^2+a^2\}^2+a^2]^2+\ldots+a^2]^2$ are always Pythagorean's n-tuples, where x_{n-2} and a are integers.

2.4. Fourth Method to Generate n-Tuples

In this method, first we will express Pythagorean's n-tuples in algebraic form. These will, then be put in relation $A_1^2 + A_2^2 + A_3^2 +.....+A_{n-1}^2 = A_n^2$ Constant terms of this relation will be equated to zero and value of variable will be found from resultant linear algebraic equation. Substitution of this value in algebraic tuple will yield the result. Assuming $A_1, A_2, A_3, ..., A_n$ to be Pythagorean's n-tuples satisfying relation $A_1^2 + A_2^2 + A_3^2 +.....+A_{n-1}^2 = A_n^2$ and a_1x, a_2x, a_3x,...,$(a_{n-1}x + b)$, $(a_nx + a)$, their algebraic representations, then

$$(a_nx + a)^2 = (a_{n-1}x + b) + (a_1x)^2 + (a_2x)^2 + ... + (a_{n-2}x)^2. \quad (25)$$

where $a_1, a_2, a_3, ..., a_n, a$ and b are integers. For eliminating constant terms, we put $a = b$, expand and simplify the equation, to obtain, $x = P/Q$, where

$P = 2a(a_n - a_{n-1})$ and

$Q = (a_1^2 + a_2^2 + a_3^2 + ... + a_{n-1}^2 - a_n^2$. Substituting this value of x in Equation (25), following n-tuples obtained after normalization. $[a_1(P/Q)], [a_2(P/Q)], [a_3(P/Q)], , , , , [a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q) + a], [a_n(P/Q) + a]$. Normalization here, means operation of multiplication with lowest common multiplier abbreviated LCM to convert tuples in fractions to integers. This derivation proves LEMMA 15.

LEMMA 15:

$[a_1(P/Q)], [a_2(P/Q)], [a_3(P/Q)], to..., [a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q) + a]$, $[a_n(P/Q) + a]$. On normalization, are always Pythagorean's n-tuples, where $P = 2a(a_n - a_{n-1})$, and $Q = (a_1^2 + a_2^2 + a_3^2 + ... + a_{n-1}^2)$ and a_1, a_2, a_3,...,a_n are real rational quantities.

On putting, $a_1 = a_2 = a_3 =...=a_{n-3} = 0$, triples obtained are, $[a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q) + a]$, $[a_n(P/Q) + a]$. This proves LEMMA 16.

LEMMA 16:

$[a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q) + a]$, $[a_n(P/Q) + a]$. on normalisation, are always Pythagorean's triples, where $P = 2a(a_n - a_{n-1})$ and $Q = (a_{n-1}^2 - a_n^2)$ and a, a_{n-2}, a_{n-1}, a_n^2 are real rational quantities.

Examples:

When $(a, a_n, a_{n-1}, a_{n-2})$ are (3, 1, 2, 3), (11, 2, 3, 5), then (P/Q) on simplification are (1/2), (-11/15) and triples, on ignoring negative signs, are (3, 4, 5), (55, 132, 143) respectively. On putting, $a_1 = a_2 = a_3 = ... = a_{n-4} = 0$, we obtain quadruples $[a_{n-3}(P/Q)]$, $[a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q)+a]$, $[a_n(P/Q)+a]$. This proves LEMMA 17

LEMMA 17:

$[a_{n-3}(P/Q)]$, $[a_{n-2}(P/Q)]$, $[a_{n-1}(P/Q) + a]$, $[a_n(P/Q) + a]$, on normalization, are always Pythagorean's quadruples, where $P = 2a(a_n - a_{n-1})$ and $Q = (a_{n-3}^2 + a_{n-2}^2 + a_{n-1}^2) - a_n^2$ and $a, a_{n-3}, a_{n-2}, a_{n-1}, a_n$ are real rational quantities.

Examples:

When $(a, a_n, a_{n-1}, a_{n-2}, a_{n-3})$ are (3, 1, 2, 3, 7), (7, 4, 2, 3, 5), then (P/Q) on simplification are (-6/61), (14/11), (-108/83) and quadruples, on ignoring negative signs, are (42, 18, 171, 177), (70, 42, 105, 133) respectively. On putting, $a_1 = a_2 = a_3 = ... = a_{n-5} = 0$, we obtain quintuples $[a_{n-4}P]$, $[a_{n-3}P]$, $[a_{n-2}P]$, $[a_{n-1}P + Qa]$, $[a_nP + Qa]$.This proves LEMMA 18.

LEMMA 18:

$[a_{n-4}P], [a_{n-3}P], [a_{n-2}P], [a_{n-1}P + Qa], [a_nP + Qa]$, on normalization, are always Pythagorean's quadruples, where,

$P = 2a(a_n - a_{n-1})$ and $Q = (a_{n-4}^2 + a_{n-3}^2 a_{n-2}^2 + a_{n-1}^2) - a_n^2$ and $a, a_{n-4}, a_{n-3}, a_{n-2}^2, a_{n-1}, a_n^2$ are rational quantities.

Examples:

When $(a, a_n, a_{n-1}, a_{n-2}, a_{n-3}, a_{n-4})$ are (5, 3, 1, 2, 3, 7), (1, 7, 4, 2, 3, 5), then (P/Q), on simplification are (10/27), (6/5) and quintuples, on ignoring negative signs, are (70, 30, 20, 145, 165), (30, 18, 12, 29, 47) respectively. In this way, we can determine n-tuples by assigning different values to a, a_1, to a_{n-2}.

3. Results and Discussion

A simple algebraic identity $(x^2 + a)^2 = x^4 + a^2 + 2x^2a$ can be transformed to Pythagorean's quadruple $(x^2 + 2a_1^2)^2 = (x^2)^2 + (2a_1^2)^2 + (2xa_1)^2$ by substituting

$a = 2a_1^2$, when x, a, a_1 are real rational quantities. If $'a'$ happens to equal $2(a_1^2+a_2^2+a_3^2+a_4^2+...+a_{n-3}^2)$, the identity transforms to Pythagorean's n-tuples, when $a_1, a_2, a_3, ..., a_{n-3}$ are real rational quantities. Also the identity $(x^2 + a^2)^2 - (x^2 - a^2)^2 = (2xa)^2$ itself is a Pythagorean's triple. Similarly, substituting a^2 with $(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2)$ or x^2 with $(x_1^2 + x_2^2 + x_3^2 + ... + x_{n-3}^2)$, the identity transforms to Pythagorean's n-tuples.

A close look at identity, $(x_1^2 + a^2)^2 = (x_1^2 - a^2)^2 + (2x_1a)^2$ reveals that it has the potential to establish a recursive relation in $(x_1^2 + a^2)$ provided $(x_1^2 + a^2)$ also appears in right hand side. This objective can be achieved if we substitute x1 appearing in term $(2x_1a)^2$ with $(x_2^2 + a_1^2)^2$, such substitution renders $\{(x_2^2 + a_2^2)^2 + a_2^2\}^2 = \{(x_2^2 + a^2)^2 - a_2^2\}^2 + \{(2a)^2(x_2^2 + a^2)^2\}$.

Now $(x_2^2+a^2)^2$ appearing in the term $\{(2a)^2(x_2^2+a^2)^2\}$ can be substituted by $(x_2^2 - a)^2 + (2x_2a)^2$ culminating into a relation, $\{(x_2^2 + a^2)^2 + a^2\}^2 = \{(x_2^2 + a^2)^2 - a_2^2\}^2 + \{(2a)(x_2^2 - a^2)^2\} + \{(2a)^2x_2\}^2$ Successive substitution leads to generation of Pythagorean's n-tuples. Another method utilised is conversion of an algebraic equation in to an identity. In this method, n-tuples are repersented in algebraic form as $a_1x, a_2x, a_3x, ..., a_{n-2}x, (a_{n-1}x+a)$ and (a_nx+a) and finding value of variable x from relation $(a_nx + a)^2 = (a_{n-1}x + a) + (a_1x)^2 + (a_2x)^2 + ... + (a_{n-2}x)$ then substituting this value of x obtained from above equation, in n-tuples in algebraic form.

Of these, the method, where recursive relation is used, yields quintuples and higher tuples which have magnitudes too large to handle owing to the fact that as we proceed from one tuple to one step higher, the values of subsequent tuples get almost squared. Therefore, methods other than this, are preferable, when tuples of low values are required.

Overview of what we have proved, concludes to the fact, expansion of $(x^2 + a)^2$ or $(x^2 + a)^2 - (x^2 + a)^2$ yields a term $(2x^2a)$ or $(4x^2a)$ containing $'a'$ which is utilised for forming a set of squares $(2xa_1)^2$, $(2xa_2)^2$, $(2xa_3)^2$, ...,$(2xa_{n-2})^2$, establishment of a recursive relation in triples can generate n-tuples and also tuples in algebraic form when put in relation $(an_x + a)^2 = (a_{n-1}x + a)^2 + (a_1x)^2 + (a_2x)^2 + (a_3x)^2 + ... + (a_{n-2}x + a)^2$,can be transformed to a linear equation leading to their parameterisation.

4. Conclusion

Self-generating n-tuples literally means those Pythagorean's tuples that generate without any external help. That means, such tuples should be by-product of common algebraic identities and should spring up at their own. Also n-tuples can be generated by writing an algebraic equation, where each term in left hand side as well as right hand side involve squares. Obtaining value of variable x from linear equation that was reduced from quadratic equation involving squares and then inserting this value in place of x in algebraic terms, yield n-tuples. Based on the identities including identity obtained from algebraic equation Pythagorean's n-tuples can be obtained from parametric solutions listed below.

First Parametric Solution

$$\{x^2 + 2(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2)\}^2 = (x^2)^2 + \{2(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-3}^2)\}^2 + (2a_1x)^2 + (2a_2x)^2 + (2a_3x)^2 + ... + (2a_{n-3}x)^2.$$

Second Parametric Solution

$$\{x^2 + 2(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-2}^2)\}^2 = [(x^2) - \{2(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-2}^2)\}^2 + (2a_1x)^2 + (2a_2x)^2 + (2a_3x)^2 + ... + (2a_{n-2}x)^2.$$

Third Parametric Solution

$$[[\{(x_{n-2}^2 + a^2)^2 + a^2\}^2 + a^2]^2 + ... + a^2]^2 = [[[\{(x_{n-2}^2 + a^2)^2 + a^2\}^2 + a^2]^2 + ... - a^2]^2 + [2a[\{(x_{n-2}^2 + a^2)^2 + a^2\}^2 + ... - a^2]]^2 + [2^2a^2\{(x_{n-2}^2 + a^2)^2 + ... - a^2\}]^2 + ... + [2^{(n-3)}a^{(n-3)}(x_{n-2}^2 - a^2)]^2 + [2^{(n-2)}a^{(n-2)}(x_{n-2})]^2.$$

Fourth Parametric Solution

$(a_nx + a)^2 = (a_{n-1}x + a) + (a_1x)^2 + (a_2x)^2 + ... + (a_{n-2}x)^2$, where,

$$x = -2a\frac{(a_{n-1} - a_n)}{(a_1^2 + a_2^2 + a_3^2 + ... + a_{n-1}^2) - a_n^2}.$$

and n-tuples $a_1x, a_2x, a_3x, ..., (a_{n-1}x + a)$ are converted to integers by multiplication with lowest common multiplier.

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Declaration

It is an original data and has neither been sent elsewhere nor published anywhere.

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References

[1] D. E. Joyce, "Book X, Proposition XXIX", Euclid's Elements, Clark University, (1997).

[2] L.E. Dickson, Some relations between the theory of numbers and other branches of mathematics in Villat (Henri), ed., General Conference, Proceedings of the International Congress of Mathematicians, Strasbourg, Toulouse, pp.41–56(1921); reprint, Nendeln/Liechtenstein: Kraus Re-print Limited, pp.579–594(1967).

[3] R. D. Carmichael, Diophantine Analysis, New York: John Wiley & Sons, 1915.

[4] R. Spira, The Diophantine equation $x^2 + y^2 + z^2 = m^2$, Amer Math. Monthly Vol.69(5), p.360,(1962). doi:10.1080/00029890.1962.11989898

[5] P. Oliverio, Self Generating Pythagorean's quadruples and n-tuples, Fibonacci Quarterly, 34(2),p.98,(1996).

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